Similar Solutions and Integral Quantities of the Rotating Compressible Laminar Boundary Layer

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Abstract

THE "Falkner-Skan type" equations are established for a compressible laminar boundary layer on a rotating body in axial flow. The coupling between the axial and circumferential flows is analyzed and the effect of rotation rate upon the velocity profiles is shown. Similar solutions are used for the polynomial representation of the integral quantities. The shape factor H is computed as an example. Finally, the independence of the integral quantities on ω is established, leading to a very useful universality of the polynomial representation.

Contents

The study of the rotating compressible laminar boundary layer is of critical importance when such phenomena as roll damping, Magnus effects, and drag are considered. Moreover, it can be considered as a particular three-dimensional flow configuration. The major importance of the similar solutions is that they are well adapted for the integral methods to calculate the boundary layer and associated flow, including separation on complex configurations. Such an "integral" method has been developed with success by Klineberg et al.[1] for the case of shock-wave/boundary-layer interaction problems. The purpose of this investigation [2] is to compute the similar solutions to establish the polynomial of the integral quantities for the rotating compressible boundary layer.

The compressible boundary-layer equations of continuity, momentum, and energy conservation are given by Mager[3] in the s (axial), y (circumferential), and z (normal to the surface) rotating coordinate system. As pointed out by Chu and Tifford,[4] the similarity of the y-momentum and energy equations leads them to a particular form in the case of the adiabatic flow, constant rotational speed Ω, and a boundary-layer thickness δ that is small with respect to the wall radius r_w:

$$\frac{T}{T_e} = 1 + \frac{\gamma - 1}{2} M_e^2 \left(1 - \frac{u^2 + v^2}{u_e^2} + \frac{2r_w\Omega}{u_e^2}\right) \tag{1}$$

The boundary-layer equations are transformed into the incompressible plane (S,Z) using the well-known Stewartson transformation. Provided that the external velocity U_e has a power form of S, and using the stream function $\psi(S,Z)$ of the variable η given by

$$\psi = \left(\nu_\infty U_e S \frac{2}{m+1}\right)^{1/2} f(\eta) \quad \text{and} \quad \eta = \left(\frac{U_e}{\nu_\infty S}\frac{m+1}{2}\right)^{1/2} Z \tag{2}$$

the Falkner-Skan equations for the boundary layer over a rotating axisymmetric body are:

$$f''' + ff'' - \beta f' + \beta[1 + \omega(2g - g^2)] = 0 \tag{3}$$

$$g'' + fg' = 0 \tag{4}$$

where β and ω are the pressure gradient and the rotating parameters, respectively.

Equations (3) and (4) are similar to those of Crabtree[5] for the boundary layer over a yawed wing. The coupling term $\omega(2g - g^2)$ in Eq. (3) is derived from the energy equation (1). It is called the "compressibility effect" by Crabtree.

Equations (3) and (4) are solved by Runge-Kutta integration with a standard shooting technique in which a succession of f''_w and g'_w values are guessed until the external boundary conditions ($f' \to 1$, $g \to 0$) are satisfied.

The effects of rotation on the accelerated ($\beta = 2$), retarded ($\beta = -0{,}1$), and flat plate ($\beta = 0$) similar velocity profiles (f' and g) are shown in Fig. 1. For zero pressure gradient Eqs. (3) and (4) are similar and independent of the rotation. Then $g = 1 - f'$ in accordance with the boundary conditions. Following Crabtree, the contribution of the coupling term to f' and g can be evaluated by linearization of these functions for the small values of ω:

$$f' = \omega^0 f'_0 + \omega^1 f'_1, \quad g = \omega^0 g_0 + \omega^1 g_1 \tag{5}$$

f' and g are introduced in Eqs. (3) and (4) to give two independent systems which are solved separately. f'_1 and g_1 are represented only for $\beta = 2$ in Fig. 1. The major effect is to increase the main velocity U and to reduce v in their central parts.

Longitudinal and circumferential velocity profiles issued from the solutions of Eqs. (3) and (4) are computed for the rotating parameter ω equal to 0.07, 0.23, and 0.64, which correspond to 20, 40, and 60,000 rpm, respectively. For the

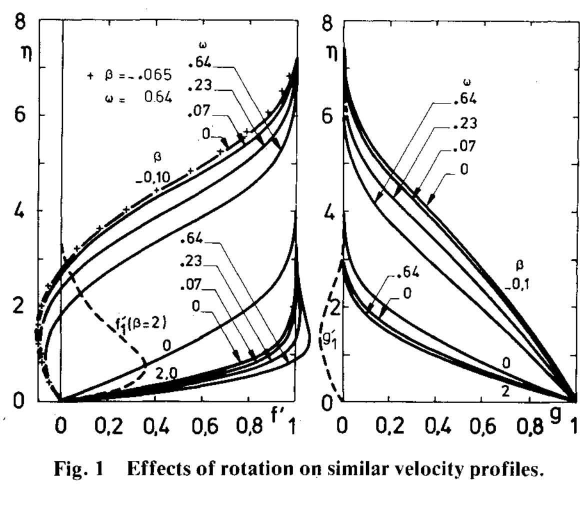

Fig. 1 Effects of rotation on similar velocity profiles.

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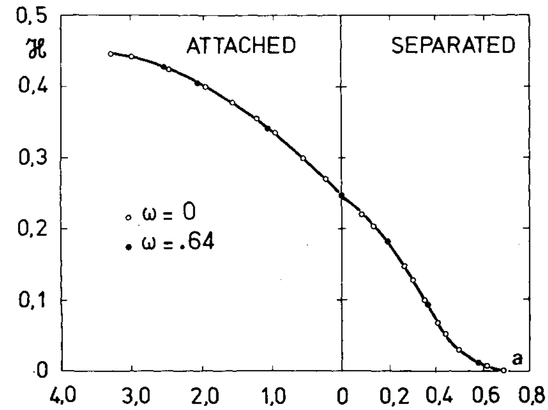

Fig. 2 Example of ω independence for integral quantities. Shape factor $H(a)$.

highest values of ω an overshoot is observed in the central part of f' for the accelerated flow. Whereas the rotation decreases the separation effects for retarded flows, the circumferential velocity components are reduced as ω increases for both attached and separated boundary layers. It can be concluded from these results that the effect of the rotation is to delay the separation of the boundary layer.

The quantities used in the integral methods for the viscous-inviscid problem resolution are computed from the similar solutions. In this paper we consider the well-known shape parameter $H=\theta_i/\delta_i^*$ which depends only upon the axial velocity U. A similar discussion could be conducted with the integral quantities derived from v and U,v.

The shape parameter is only a function of $f'(\eta)$:

$$H=\int_0^{\eta_{0.99}} (1-f')f'\,\mathrm{d}\eta\Big/\int_0^{\eta_{0.99}} (1-f')\,\mathrm{d}\eta \qquad (6)$$

According to Lees and Reeves, H can be expressed as a polynomial of a similar incompressible velocity profile parameter a. They chose two different definitions for a, depending on the attached or separated nature of the boundary layer:

$$a_{\mathrm{att}}=\left|\frac{\dfrac{\partial(U/U_e)}{\partial}}{\partial(Z/\delta_i)}\right|_w=\eta_{0.99}\,f''_w$$

$$a_{\mathrm{sep}}=Z(U/U_e=0)/\delta_i=\eta_{f'=0}/\eta_{0.99} \qquad (7)$$

a is related to β in the two cases. A value of a, and therefore a value of H corresponds to each family of similar velocity profiles. It is then easy to construct the polynomial $H(a)$ for many of attached and separated boundary-layer velocity profiles.

The solution by the integral method of a viscous-inviscid problem as the shock-wave/boundary-layer interaction on a rotating body requires the construction of many polynomials for each value of ω, which is an important penalty for the method. Therefore it would be fruitful to establish the independence of the polynomials for ω. This independence will exist if a unique value of each integral quantity can be found for a pair of (β,ω) values. Then the same velocity profiles correspond to $(\beta,\omega)_1$ and $(\beta,\omega)_2$. A particular coupling is the nonrotating case β_{NR} for $\omega=0$. Therefore it is suitable to establish a general relation between $(\beta,\omega)_n$ and β_{NR} which allows the uniqueness of the polynomials.

Horton showed that the f' and g functions can be linearized for the small values of β, as was done previously for ω. Then a linear relation exists between β and ω which can be written:

$$\beta_{\mathrm{NR}}/\beta=1+\omega \qquad (8)$$

In Fig. 1, the crosses representing $\beta=-0.10$ correspond to the velocity profile such that relation (8) is satisfied. The correspondence between the nonrotating profile ($\beta=-0.10$, $\omega=0$) and the rotating one ($\beta=-0.065$, $\omega=0.64$) is excellent.

A systematic computation of velocity profiles U/U_e has been undertaken, creating many pairs of (β,ω) values such that the a value is held constant for attached and separated flow. The results show a linear behavior for the β,ω parameter variation:

$$\beta_{\mathrm{NR}}/\beta=K(\beta_{\mathrm{NR}})(1+\omega) \qquad (9)$$

with $K(\beta_{\mathrm{NR}})$ between 0.75 and 1. Then for every pair of (β,ω) values there is a corresponding value of β_{NR} such that the relevant velocity profiles have the same a and consequently the same polynomials.

This important property is verified in Fig. 2, where $H(a)$ is computed for both $\omega=0.0045$ and 0.64. The independence of the other integral functions of a, b, and (a,b) on ω is also established and the very useful universality of the polynomial representation is then demonstrated.

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References

[1]Klineberg, J.M., Kubota, T., and Lees, L., "Theory of Exhaust-Plume/Boundary-Layer Interactions at Supersonic Speeds," *AIAA Journal*, Vol. 10, May 1972, pp. 581-588.

[2]Leblanc, R., "Profils semblables de vitesse et quantités intégrales d'une couche limite sur cylindre tournant," von Kármán Institute, Rhode St. Génese, Belgium, TN 768, Nov. 1982.

[3]Mager, A., "Three Dimensional Laminar Boundary-Layers," *Theory of Laminar Flows*, Princeton Series, Vol. IV, edited by F.K. Moore, Princeton University Press, Princeton, N.J. 1964.

[4]Chu, S.T. and Tifford, A.N., "The Compressible Laminar Boundary-Layer on a Rotating Body of Revolution," *Journal of the Aerospace Sciences*, Vol. 21, May 1954, pp.345-346.

[5]Crabtree, L.F."The Compressible Laminar Boundary Layer on a Yawed Infinte Wing,"*Aeronautical Quarterly*, Vol. 5., 1954.

[6]Horton, H.P., private communication.